

FACTORIZATION IN SL^∞

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ABSTRACT. We show that the non-separable Banach space SL^∞ is primary. This is achieved by directly solving the infinite dimensional factorization problem in SL^∞ . In particular, we bypass Bourgain's localization method.

1. INTRODUCTION

Let \mathcal{D} denote the collection of dyadic subintervals of the unit interval $[0, 1)$, and let h_I denote the L^∞ -normalized Haar function supported on $I \in \mathcal{D}$; that is, h_I is $+1$ on the left half of I , h_I is -1 on the right half of I , and zero otherwise.

The *non-separable* Banach space SL^∞ is the linear space

$$\{f = \sum_{I \in \mathcal{D}} a_I h_I : \|f\|_{SL^\infty} < \infty\}, \quad (1.1)$$

equipped with the norm

$$\left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{SL^\infty} = \left\| \left(\sum_{I \in \mathcal{D}} a_I^2 h_I^2 \right)^{1/2} \right\|_{L^\infty}. \quad (1.2)$$

We want to emphasize that throughout this paper, whenever we encounter infinite sums in the Banach space SL^∞ , we treat these series as a formal series representing the vector of coefficients, and we do *not* imply any kind of convergence. The *Hardy space* H^1 is the completion of

$$\text{span}\{h_I : I \in \mathcal{D}\}$$

under the norm

$$\|f\|_{H^1} = \int_0^1 \left(\sum_{I \in \mathcal{D}} |a_I|^2 h_I^2(x) \right)^{1/2} dx, \quad (1.3)$$

where $f = \sum_{I \in \mathcal{D}} a_I h_I$. We note the well-known and obvious inequality (see e.g. [13]):

$$|\langle f, g \rangle| \leq \|f\|_{SL^\infty} \|g\|_{H^1}, \quad f \in SL^\infty, \quad g \in H^1. \quad (1.4)$$

Let T denote a bounded, linear operator on SL^∞ . We say an operator T has *large diagonal* with respect to the Haar system $(h_I : I \in \mathcal{D})$ if there exists a $\delta > 0$ such that

$$|\langle Th_I, h_I \rangle| \geq \delta |I|, \quad I \in \mathcal{D}.$$

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2. MAIN RESULTS

The first result Theorem 2.1 asserts that the identity operator on SL^∞ factors through any operator on SL^∞ that has large diagonal with respect to the Haar system.

Theorem 2.1. *Let $\delta, \eta > 0$, and let $T : SL^\infty \rightarrow SL^\infty$ be an operator satisfying*

$$|\langle Th_I, h_I \rangle| \geq \delta |I|, \quad I \in \mathcal{D}.$$

Then the identity operator Id on SL^∞ factors through T , that is, there are operators $R, S : SL^\infty \rightarrow SL^\infty$ such that the diagram

$$\begin{array}{ccc} SL^\infty & \xrightarrow{\text{Id}} & SL^\infty \\ R \downarrow & & \uparrow S \\ SL^\infty & \xrightarrow{T} & SL^\infty \end{array} \quad (2.1)$$

is commutative. Moreover, the operators R and S can be chosen with $\|R\|\|S\| \leq (1 + \eta)/\delta$.

Let us now recall the notion of a primary Banach space, see e.g. [20]: A Banach space X is *primary* if for every bounded projection $Q : X \rightarrow X$, either $Q(X)$ or $(\text{Id} - Q)(X)$ is isomorphic to X .

The subsequent factorization result Theorem 2.2 follows from Theorem 2.1 by means of well established combinatorics of dyadic intervals, see e.g. [24]. Note that Theorem 2.1 is a theorem about an operator and a basis, whereas Theorem 2.2 expresses an isomorphic invariant.

Theorem 2.2. *Let $T : SL^\infty \rightarrow SL^\infty$ be a bounded linear operator and $\eta > 0$. Then the identity operator Id on SL^∞ factors through $H = T$ or $H = \text{Id} - T$, i.e., there exist operators $R, S : SL^\infty \rightarrow SL^\infty$ such that the diagram*

$$\begin{array}{ccc} SL^\infty & \xrightarrow{\text{Id}} & SL^\infty \\ R \downarrow & & \uparrow S \\ SL^\infty & \xrightarrow{H} & SL^\infty \end{array} \quad (2.2)$$

is commutative. Moreover, the operators R and S can be chosen with $\|R\|\|S\| \leq 2 + \eta$. Consequently, the Banach space SL^∞ is primary.

Historically, the method used to prove factorization theorems or the primarity of separable Banach spaces (e.g. [10, 21, 9, 1, 11, 14, 5, 6, 7, 8, 23, 16]) has been based on infinite dimensional reasoning, whereas the method used in non-separable Banach spaces (e.g. [4, 22, 3, 2, 26, 27, 25, 18, 17]) was finite dimensional in nature. The localization method used for non-separable spaces goes back to Bourgain [4]. There is but one exception where the reasoning in a non-separable Banach space is infinite dimensional: Lindenstrauss [19] showing that ℓ^∞ is prime. Recall that an infinite dimensional Banach space X is *prime* if every infinite dimensional complemented subspace is isomorphic to X , see e.g. [20].

The key point of this paper is that using Bourgain's localization method in non-separable Banach spaces is not a naturally occurring necessity. Specifically, we prove Theorem 2.1 and Theorem 2.2 using just infinite dimensional methods.

3. BLOCK BASES AND PROJECTIONS IN SL^∞

Here, we specify the conditions (J1)–(J4) (which go back to Jones [15]) under which a block basis of the Haar system in SL^∞ spans a complemented copy of SL^∞ . We also show that the conditions (J1)–(J4) are stable under reiteration.

3.1. Jones' compatibility conditions for SL^∞ .

Let $\mathcal{J} \subset \mathcal{D}$ be a collection of dyadic index intervals and let \mathcal{N} be a collection of sets. For all $I \in \mathcal{J}$ let $\mathcal{B}_I \subset \mathcal{N}$. We define

$$\mathcal{B} = \bigcup_{I \in \mathcal{J}} \mathcal{B}_I \quad \text{and} \quad B_I = \bigcup_{N \in \mathcal{B}_I} N, \quad \text{for all } I \in \mathcal{J}. \quad (3.1)$$

We say that the sequence $(\mathcal{B}_I : I \in \mathcal{J})$ satisfies Jones' compatibility conditions (see [15]) with constant κ_J , if the following conditions (J1)–(J4) are satisfied.

- (J1) The collection \mathcal{B} consists of finitely many measurable and nested sets of positive measure.
- (J2) For each $I \in \mathcal{J}$, the collection \mathcal{B}_I is non-empty and consists of pairwise disjoint sets. Furthermore, $\mathcal{B}_{I_0} \cap \mathcal{B}_{I_1} = \emptyset$, whenever $I_0, I_1 \in \mathcal{J}$ are distinct.
- (J3) For all $I_0, I_1 \in \mathcal{J}$ holds that

$$B_{I_0} \cap B_{I_1} = \emptyset \text{ if } I_0 \cap I_1 = \emptyset, \quad \text{and} \quad B_{I_0} \subset B_{I_1} \text{ if } I_0 \subset I_1.$$

- (J4) For all $I_0, I \in \mathcal{D}$ with $I_0 \subset I$ and $N \in \mathcal{B}_I$, we have

$$\frac{|N \cap B_{I_0}|}{|N|} \geq \kappa_J^{-1} \frac{|B_{I_0}|}{|B_I|}.$$

For a discussion on the conditions (J1)–(J4) and Jones' conditions [15] in BMO see Remark 3.4.

In the following Lemma 3.1, we record three facts about collections satisfying (J1)–(J4).

Lemma 3.1. *Let $(\mathcal{B}_I : I \in \mathcal{D})$ satisfy (J1)–(J4). Then the following statements are true:*

- (i) $(B_I : I \in \mathcal{D})$ is a sequence of nested measurable sets of positive measure.
- (ii) Let $I, I_0 \in \mathcal{D}$, then

$$B_{I_0} \subset B_I \quad \text{if and only if} \quad I_0 \subset I.$$

- (iii) Let $I_0, I \in \mathcal{D}$, with $I_0 \subset I$. Then for all $N_0 \in \mathcal{B}_{I_0}$ there exists a set $N \in \mathcal{B}_I$ such that $N_0 \subset N$.

Proof. Since B_I is a finite union of measurable sets having positive measure, we only need to show the nestedness. Let $I_0, I_1 \in \mathcal{D}$ be such that $B_{I_0} \cap B_{I_1} \neq \emptyset$. If we assumed that the intersection $I_0 \cap I_1$ were empty, then by (J3) we arrive at the contradiction $B_{I_0} \cap B_{I_1} = \emptyset$. Hence, we now know that $I_0 \cap I_1 \neq \emptyset$, which certainly implies that $I_0 \subset I_1$ or $I_1 \subset I_0$. Using (J3) concludes the proof of (i).

One of the implication of (ii) follows from (J3). We will now show the other one. To this end let $I, I_0 \in \mathcal{D}$ and $B_{I_0} \subset B_I$ and assume that $I_0 \not\subset I$. If $I_0 \cap I = \emptyset$, then $B_{I_0} = B_{I_0} \cap B_I = \emptyset$ by (J3), which contradicts (J1). Thus we know $I_0 \supsetneq I$, and so we can find a $J \in \mathcal{D}$ with $J \cap I = \emptyset$ and $I \cup J \subset I_0$. Hence, (J3) yields $B_I \cup B_J \subset B_{I_0}$, which combined with our hypothesis $B_{I_0} \subset B_I$ gives us $B_J = \emptyset$, which contradicts (J1).

Finally, we will show (iii). Suppose that (iii) is false. Then there exists $N_0 \in \mathcal{B}_{I_0}$ such that

$$N_0 \not\subset N, \quad N \in \mathcal{B}_I. \quad (3.2)$$

By (J3) we have $B_{I_0} \subset B_I$, thus we know that there exists an $N \in \mathcal{B}_I$ such that $N_0 \cap N \neq \emptyset$. Therefore, we obtain from (3.2) and the nestedness of the collection \mathcal{N} that $N \subset N_0$. But then (J3) gives us

$$N \cap B_{I_1} \subset B_{I_0} \cap B_{I_1} = \emptyset. \quad (3.3)$$

By (J4) and (J1) we obtain that

$$\frac{|N \cap B_{I_1}|}{|N|} \geq \kappa_J^{-1} \frac{|B_{I_0}|}{|B_I|} > 0.$$

The latter inequality contradicts (3.3). \square

3.2. Reiterating Jones' compatibility conditions.

Jones' compatibility conditions (J1)–(J4) are stable under iteration in the following sense.

Theorem 3.2. *Let $(\mathcal{A}_I : I \in \mathcal{D})$ be a sequence of collections of sets that satisfies (J1)–(J4) with constant κ_J . Put $\mathcal{M} = \bigcup_{I \in \mathcal{D}} \mathcal{A}_I$ and $A_I = \bigcup_{M \in \mathcal{A}_I} M$. Let \mathcal{N} denote the collection of nested sets given by*

$$\mathcal{N} = \{A_I : I \in \mathcal{D}\}.$$

For each $J \in \mathcal{D}$ let $\mathcal{B}_J \subset \mathcal{N}$ be such that $(\mathcal{B}_J : J \in \mathcal{D})$ satisfies (J1)–(J4) with constant κ_J , where we put $B_J = \bigcup_{A_I \in \mathcal{B}_J} A_I$. Finally, for all $J \in \mathcal{D}$, we define

$$\mathcal{C}_J = \bigcup_{A_I \in \mathcal{B}_J} \mathcal{A}_I \quad \text{and} \quad C_J = \bigcup_{A_I \in \mathcal{B}_J} A_I = B_J.$$

Then $(\mathcal{C}_J : J \in \mathcal{D})$ is a sequence of collections of sets in \mathcal{M} satisfying (J1)–(J4) with constant κ_J^2 .

Proof. By (J3) and Lemma 3.1 (ii) for (\mathcal{A}_I) we obtain that \mathcal{N} consists indeed of nested sets. Since $C_J = B_J$, it is clear that (\mathcal{C}_J) satisfies (J3) and (J1).

We will now show that (C_J) satisfies (J2). To this end, let $M_0, M_1 \in \mathcal{C}_J$ and assume that $M_0 \cap M_1 \neq \emptyset$. Per definition of \mathcal{C}_J , there exist $I_0, I_1 \in \mathcal{D}$ such that $A_{I_0}, A_{I_1} \in \mathcal{B}_J$ and $M_i \in \mathcal{A}_{I_i}$, $i = 0, 1$. This implies $A_{I_0} \cap A_{I_1} \neq \emptyset$, so by the first part of (J2) for (\mathcal{B}_J) we obtain $I_0 = I_1$. Hence, $M_0, M_1 \in \mathcal{A}_{I_0}$, and the second part of (J2) for (\mathcal{A}_I) yields $M_0 = M_1$.

Next, we verify that (\mathcal{C}_J) satisfies (J4). Let $J_0, J \in \mathcal{D}$ with $J_0 \subset J$ and let $M \in \mathcal{C}_J$. We need to show that

$$\frac{|M \cap C_{J_0}|}{|M|} \geq \kappa_J^{-2} \frac{|C_{J_0}|}{|C_J|}.$$

Per definition of \mathcal{C}_J , there exists a dyadic interval I so that $A_I \in \mathcal{B}_J$ and $M \in \mathcal{A}_I$. Property (J4) for the collection (\mathcal{B}_J) and the definition of B_{J_0} give

$$\frac{|C_{J_0}|}{|C_J|} = \frac{|B_{J_0}|}{|B_J|} \leq \kappa_J \frac{|A_I \cap B_{J_0}|}{|A_I|} = \kappa_J \sum_{A_{I_0} \in \mathcal{B}_{J_0}} \frac{|A_I \cap A_{I_0}|}{|A_I|}.$$

Whenever $A_I \cap A_{I_0} \neq \emptyset$, Lemma 3.1 (iii) applied to (\mathcal{B}_J) yields that $A_{I_0} \subset A_I$. By Lemma 3.1 (ii) applied to (\mathcal{A}_J) , we obtain that $A_{I_0} \subset A_I$ is equivalent to $I_0 \subset I$. Thus we note

$$\frac{|C_{J_0}|}{|C_J|} \leq \kappa_J \sum_{\substack{I_0 \subset I \\ A_{I_0} \in \mathcal{B}_{J_0}}} \frac{|A_{I_0}|}{|A_I|}. \quad (3.4)$$

Condition (J4) for the collection (\mathcal{A}_I) and the definition of C_{J_0} give

$$\sum_{\substack{I_0 \subset I \\ A_{I_0} \in \mathcal{B}_{J_0}}} \frac{|A_{I_0}|}{|A_I|} \leq \kappa_J \sum_{\substack{I_0 \subset I \\ A_{I_0} \in \mathcal{B}_{J_0}}} \frac{|M \cap A_{I_0}|}{|M|} \leq \kappa_J \frac{|M \cap C_{J_0}|}{|M|}. \quad (3.5)$$

Combining (3.4) and (3.5) concludes the proof. \square

3.3. Embeddings and projections in SL^∞ .

Here we establish that if $(\mathcal{B}_I : I \in \mathcal{D})$ satisfies Jones' compatibility conditions (J1)–(J4), then the block basis $(b_I : I \in \mathcal{D})$ of the Haar system $(h_I : I \in \mathcal{D})$ be given by

$$b_I = \sum_{K \in \mathcal{B}_I} h_K, \quad I \in \mathcal{D},$$

spans a complemented copy of SL^∞ .

Theorem 3.3. *Let $\mathcal{J} \subset \mathcal{D}$ be a collection of index intervals, and let $\mathcal{B}_I \subset \mathcal{D}$, $I \in \mathcal{J}$. Assume that the sequence of collections of dyadic intervals $(\mathcal{B}_I : I \in \mathcal{J})$ satisfies Jones' compatibility conditions (J1)–(J4) with constant $\kappa_J > 0$. Let the block basis $(b_I : I \in \mathcal{J})$ of the Haar system $(h_I : I \in \mathcal{D})$ be given by*

$$b_I = \sum_{K \in \mathcal{B}_I} h_K, \quad I \in \mathcal{J}. \quad (3.6)$$

Then the operators $B, Q : SL^\infty \rightarrow SL^\infty$ given by

$$Bf = \sum_{I \in \mathcal{J}} \frac{\langle f, h_I \rangle}{\|h_I\|_2^2} b_I \quad \text{and} \quad Qf = \sum_{I \in \mathcal{J}} \frac{\langle f, b_I \rangle}{\|b_I\|_2^2} h_I \quad (3.7)$$

satisfy the estimates

$$\|Bf\|_{SL^\infty} \leq \|f\|_{SL^\infty} \quad \text{and} \quad \|Qf\|_{SL^\infty} \leq \kappa_J^{1/2} \|f\|_{SL^\infty}, \quad (3.8)$$

for all $f \in SL^\infty$. Moreover, the diagram

$$\begin{array}{ccc} SL^\infty & \xrightarrow{\text{Id}_{SL^\infty}} & SL^\infty \\ & \searrow B & \nearrow Q \\ & SL^\infty & \end{array} \quad (3.9)$$

is commutative. Consequently, the range of B is complemented, and B is an isomorphism onto its range.

Proof. First, we will show the estimate for B . To this end, let $f \in SL^\infty$ be finitely supported with respect to the Haar system $(h_I : I \in \mathcal{D})$, i.e.

$$f = \sum_{I \in \mathcal{D}^{N_0}} a_I h_I,$$

for some integer N_0 and scalars $(a_I : I \in \mathcal{D}^{N_0})$. By (J2), we have

$$\|Bf\|_{SL^\infty}^2 = \sup_{x \in [0,1]} \sum_{I \in \mathcal{J} \cap \mathcal{D}^{N_0}} a_I^2 \mathbb{1}_{B_I}(x). \quad (3.10)$$

Given $x \in [0,1]$, we define $I(x) \in \mathcal{D}^{N_0} \cup \{\emptyset\}$ by

$$I(x) = \bigcap \{J : J \in \mathcal{J} \cap \mathcal{D}^{N_0}, B_J \ni x\}.$$

By definition of $I(x)$, we have that $I \supset I(x)$, whenever $B_I \ni x$. Thus, the following inequalities hold:

$$\sum_{I \in \mathcal{J} \cap \mathcal{D}^{N_0}} a_I^2 \mathbb{1}_{B_I}(x) \leq \sum_{\substack{I \in \mathcal{J} \\ I \supset I(x)}} a_I^2 \mathbb{1}_I(y) \leq \sum_{I \in \mathcal{J}} a_I^2 \mathbb{1}_I(y), \quad y \in I(x).$$

Taking the supremum over all $x \in [0,1]$ in the latter estimate yields in combination with (3.10) that

$$\|Bf\|_{SL^\infty}^2 \leq \sup_{x \in [0,1]} \inf_{y \in I(x)} \sum_{I \in \mathcal{J}} a_I^2 \mathbb{1}_I(y) \leq \|f\|_{SL^\infty}^2, \quad (3.11)$$

for all finitely supported $f \in SL^\infty$. To show the above estimate for arbitrary $f \in SL^\infty$, consider the following. Let $f \in SL^\infty$ and define f^{N_0} , by

$$f^{N_0} = \sum_{I \in \mathcal{D}^{N_0}} \frac{\langle f, h_I \rangle}{\|h_I\|_2^2} h_I, \quad N_0 \in \mathbb{N}.$$

Observe that by definition of the norm $\|\cdot\|_{SL^\infty}$ and by (3.11) we obtain

$$\|Bf\|_{SL^\infty} = \sup_{N_0 \in \mathbb{N}} \|Bf^{N_0}\|_{SL^\infty} \leq \sup_{N_0 \in \mathbb{N}} \|f^{N_0}\|_{SL^\infty} = \|f\|_{SL^\infty}.$$

Before we continue with the estimate for Q , we will now introduce some notation. Given $N_0 \in \mathbb{N}$ let \mathcal{B}_{N_0} denote the collection of building blocks given by

$$\mathcal{B}_{N_0} = \{K_0 \in \mathcal{B}_{I_0} : I_0 \in \mathcal{I} \cap \mathcal{D}_{N_0}\}.$$

Accordingly, we define the building blocks \mathcal{B}^{N_0} by

$$\mathcal{B}^{N_0} = \{K \in \mathcal{B}_I : I \in \mathcal{I} \cap \mathcal{D}^{N_0}\}.$$

We will now estimate Q .

To begin with, let us assume that f is of the following form:

$$f = \sum_{K \in \mathcal{B}^{N_0}} a_K h_K.$$

On the one hand, a straightforward calculation using only the properties of dyadic intervals shows

$$\begin{aligned} \|Qf\|_{SL^\infty}^2 &= \sup_{x \in [0,1)} \sum_{I \in \mathcal{I} \cap \mathcal{D}^{N_0}} \frac{\langle f, b_I \rangle^2}{\|b_I\|^2} \mathbb{1}_I(x) \\ &= \sup_{x \in [0,1)} \sum_{\substack{I \in \mathcal{I} \cap \mathcal{D}^{N_0} \\ I \ni x}} \left(\sum_{K \in \mathcal{B}_I} a_K \frac{|K|}{|B_I|} \right)^2 \\ &= \max_{I_0 \in \mathcal{I} \cap \mathcal{D}_{N_0}} \sum_{\substack{I \in \mathcal{I} \\ I \supset I_0}} \left(\sum_{K \in \mathcal{B}_I} a_K \frac{|K|}{|B_I|} \right)^2. \end{aligned}$$

Applying the Cauchy-Schwarz inequality to the inner sum yields

$$\|Qf\|_{SL^\infty}^2 \leq \max_{I_0 \in \mathcal{I} \cap \mathcal{D}_{N_0}} \sum_{\substack{I \in \mathcal{I} \\ I \supset I_0}} \sum_{K \in \mathcal{B}_I} a_K^2 \frac{|K|}{|B_I|}. \quad (3.12)$$

On the other hand, by the definition of $\|\cdot\|_{SL^\infty}$ and by (J2) we obtain

$$\|f\|_{SL^\infty}^2 = \sup_{x \in [0,1)} \sum_{K \in \mathcal{B}^{N_0}} a_K^2 \mathbb{1}_K(x) = \sup_{x \in [0,1)} \sum_{I \in \mathcal{I} \cap \mathcal{D}^{N_0}} \sum_{K \in \mathcal{B}_I} a_K^2 \mathbb{1}_K(x).$$

By (J3) and (J1), the collections B_{I_0} , $I_0 \in \mathcal{I} \cap \mathcal{D}_{N_0}$ are pairwise disjoint and of positive measure, hence we obtain for all measurable, non-negative functions g that

$$\sup_{x \in [0,1)} g(x) \geq \sup_{x \in [0,1)} \sum_{I_0 \in \mathcal{I} \cap \mathcal{D}_{N_0}} \left(\frac{1}{|B_{I_0}|} \int_{B_{I_0}} g(y) dy \right) \mathbb{1}_{B_{I_0}}(x).$$

Using the latter estimate for $g = \sum_{I \in \mathcal{I} \cap \mathcal{D}^{N_0}} \sum_{K \in \mathcal{B}_I} a_K^2 \mathbb{1}_K(x)$ yields

$$\|f\|_{SL^\infty}^2 \geq \sup_{x \in [0,1)} \sum_{I_0 \in \mathcal{I} \cap \mathcal{D}_{N_0}} \sum_{I \in \mathcal{I} \cap \mathcal{D}^{N_0}} \sum_{K \in \mathcal{B}_I} a_K^2 \frac{|K \cap B_{I_0}|}{|B_{I_0}|} \mathbb{1}_{B_{I_0}}(x).$$

By (J3) and Lemma 3.1 (ii) we obtain for all $K \in \mathcal{B}_I$ that whenever $K \cap B_{I_0} \neq \emptyset$, we have that $I \supset I_0$ and $B_I \supset B_{I_0}$. Therefore and by (J3), we get

$$\begin{aligned} \|f\|_{SL^\infty}^2 &\geq \sup_{x \in [0,1)} \sum_{I_0 \in \mathcal{I} \cap \mathcal{D}_{N_0}} \sum_{\substack{I \in \mathcal{I} \\ I \supset I_0}} \sum_{K \in \mathcal{B}_I} a_K^2 \frac{|K \cap B_{I_0}|}{|B_{I_0}|} \mathbb{1}_{B_{I_0}}(x) \\ &= \max_{I_0 \in \mathcal{I} \cap \mathcal{D}_{N_0}} \sum_{\substack{I \in \mathcal{I} \\ I \supset I_0}} \sum_{K \in \mathcal{B}_I} a_K^2 \frac{|K \cap B_{I_0}|}{|B_{I_0}|}. \end{aligned}$$

Finally, using (J4) yields the following lower estimate:

$$\|f\|_{SL^\infty}^2 \geq \kappa_J^{-1} \max_{I_0 \in \mathcal{I} \cap \mathcal{D}_{N_0}} \sum_{\substack{I \in \mathcal{I} \\ I \supset I_0}} \sum_{K \in \mathcal{B}_I} a_K^2 \frac{|K|}{|B_I|}.$$

Comparing the above estimate with (3.12) shows

$$\|Qf\|_{SL^\infty} \leq \kappa_J^{1/2} \|f\|_{SL^\infty}, \quad (3.13)$$

for all finitely supported $f \in SL^\infty$. Now let $\mathcal{B} = \bigcup_{N_0 \in \mathbb{N}} \mathcal{B}^{N_0}$,

$$f = \sum_{K \in \mathcal{B}} a_K h_K \quad \text{and} \quad f^{N_0} = \sum_{K \in \mathcal{B}^{N_0}} a_K h_K.$$

Note the identity

$$Qf^{N_0} = \sum_{I \in \mathcal{I} \cap \mathcal{D}_{N_0}} \frac{\langle f, b_I \rangle}{\|b_I\|_2^2} h_I.$$

The above identity, the definition of $\|\cdot\|_{SL^\infty}$ and inequality (3.13) yield

$$\|Qf\|_{SL^\infty} = \sup_{N_0 \in \mathbb{N}} \|Qf^{N_0}\|_{SL^\infty} \leq \kappa_J^{1/2} \sup_{N_0 \in \mathbb{N}} \|f^{N_0}\|_{SL^\infty} \leq \kappa_J^{1/2} \|f\|_{SL^\infty}.$$

□

Remark 3.4. The conditions (J1)–(J4) go back to Jones [15]. In [15], Jones treated projections in BMO with the following three conditions:

Suppose that $\mathcal{C}_1, \dots, \mathcal{C}_N$ are disjoint collections of intervals which satisfy

([15], 2.1) $\|\sum_{I \in \mathcal{C}_j} \mathbb{1}_I\|_{L^\infty} = 1$, $1 \leq j \leq N$,

([15], 2.2) and suppose that there are constants a_{jk} such that whenever $I \in \mathcal{C}_j$,

$$\frac{1}{2} a_{jk} \leq \frac{1}{|I|} \sum_{\substack{J \subset I \\ J \in \mathcal{C}_k}} |J| \leq a_{jk}, \quad 1 \leq j, k \leq N.$$

([15], 2.3) Furthermore we suppose whenever $2 \leq j \leq N$ and $I \in \mathcal{C}_j$ there is $J \in \mathcal{C}_{j-1}$ such that $I \subset J$.

We remark that Jones' conditions ([15], 2.1), ([15], 2.2), and ([15], 2.3) imply (J1)–(J4) (with a reasonable interpretation of the binary tree structure). Most noteworthy are the following observations:

- ▷ Condition ([15], 2.1) together with the disjointness of the collections \mathcal{C}_j (the line above ([15], 2.1)) is exactly (J2).
- ▷ The absence of the corresponding upper estimate of ([15], 2.2) in (J4).
- ▷ The “uniform packing” condition ([15], 2.2) together with the “stacking” condition ([15], 2.3) imply condition (J1) and condition (J4).
- ▷ The absence of a corresponding “stacking” condition in (J1)–(J4).

- ▷ The conditions (J1)–(J4) imply a partially ordered (with respect to inclusion) variant of Jones’ “stacking” condition, if $\mathcal{I} = \mathcal{D}$. However, that is *not* the case if \mathcal{I} is linearly ordered with respect to inclusion. We refer the reader to the proof of Lemma 3.1 (iii)).

Remark 3.5. Let $(\mathcal{B}_I : I \in \mathcal{D})$ denote a sequence of collections of dyadic intervals satisfying Jones’ compatibility conditions (J1)–(J4). Given a sequence of signs $\varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{D})$ we define

$$b_I^{(\varepsilon)} = \sum_{K \in \mathcal{B}_I} \varepsilon_K h_K. \quad (3.14)$$

We call $(b_I^{(\varepsilon)})$ the *block basis generated by* $(\mathcal{B}_I : I \in \mathcal{D})$ and ε .

The block basis $(b_I^{(\varepsilon)})$ gives rise to the operators $B^{(\varepsilon)}, Q^{(\varepsilon)} : SL^\infty \rightarrow SL^\infty$:

$$B^{(\varepsilon)} f = \sum_{I \in \mathcal{D}} \frac{\langle f, h_I \rangle}{\|h_I\|_2^2} b_I^{(\varepsilon)} \quad \text{and} \quad Q^{(\varepsilon)} f = \sum_{I \in \mathcal{D}} \frac{\langle f, b_I^{(\varepsilon)} \rangle}{\|b_I^{(\varepsilon)}\|_2^2} h_I. \quad (3.15)$$

See Theorem 3.3 for a definition of the operators B, Q . By the 1-unconditionality of the Haar system in SL^∞ and

$$Q^{(\varepsilon)} f = Q f^{(\varepsilon)}, \quad \text{where} \quad f^{(\varepsilon)} = \sum_{K \in \mathcal{D}} \varepsilon_K \frac{\langle f, h_K \rangle}{\|h_K\|_2^2} h_K,$$

we obtain the estimates

$$\|B^{(\varepsilon)} f\|_{SL^\infty} \leq \|B f\|_{SL^\infty} \quad \text{and} \quad \|Q^{(\varepsilon)} f\|_{SL^\infty} \leq \|Q f\|_{SL^\infty}, \quad (3.16)$$

for all $f \in SL^\infty$. Moreover, they have the identity

$$Q^{(\varepsilon)} B^{(\varepsilon)} = \text{Id}_{SL^\infty}. \quad (3.17)$$

Consequently, the range of $B^{(\varepsilon)}$ is complemented, and $B^{(\varepsilon)}$ is an isomorphism onto its range.

4. FACTORIZATION OF THE IDENTITY OPERATOR ON SL^∞ THROUGH OPERATORS WITH LARGE DIAGONAL

Here, we will develop the crucial tools that permit us to almost-diagonalize a given operator T on SL^∞ , see Theorem 4.3. The almost-diagonalization result Theorem 4.3, will then be used to show our first main result Theorem 2.1: we prove that the identity operator on SL^∞ factors through operators T acting on SL^∞ which have large diagonal with respect to the Haar system. By well established methods, we therefore obtain that SL^∞ is primary, which proves our second main result Theorem 2.2.

We emphasize that all of our proofs bypass Bourgain’s localization method, which has been used many times for showing the primarity of non-separable Banach spaces, see e.g. [4, 22, 3, 2, 26, 27, 25, 18, 17]. Lemma 4.2 is the key ingredient which allows us to use infinite dimensional reasoning in the *non-separable* space SL^∞ . An ℓ^∞ variant of Lemma 4.2 was used by Lindenstrauss to prove that ℓ^∞ is prime [19].

4.1. Almost-annihilating subspaces of H^1 and SL^∞ .

Firstly, we prove that Rademacher functions r_m converge to 0, when tested against functions $f \in SL^\infty$. Secondly, we show how to select large subsets of the dyadic intervals, so that a given operator $T : SL^\infty \rightarrow SL^\infty$ is small when acting on the subspace spanned by these intervals (and is tested against a function in H^1).

For any sequence of scalars $c = (c_I : I \in \mathcal{D})$, the Rademacher type function $r_m^{(c)}$ is given by

$$r_m^{(c)} = \sum_{I \in \mathcal{D}_m} c_I h_I, \quad m \in \mathbb{N}. \quad (4.1)$$

Lemma 4.1. *Let $f \in SL^\infty$ and $g \in H^1$. Then*

$$\sup_{\|c\|_{\ell^\infty} \leq 1} |\langle f, r_m^{(c)} \rangle| \rightarrow 0 \quad \text{and} \quad \sup_{\|c\|_{\ell^\infty} \leq 1} |\langle Tr_m^{(c)}, g \rangle| \rightarrow 0, \quad \text{as } m \rightarrow \infty. \quad (4.2)$$

Proof. Let $f \in SL^\infty$ and $g \in H^1$. Note that there are sequences of scalars $\theta = (\theta_I : I \in \mathcal{D})$ and $\varepsilon = (\varepsilon_I : I \in \mathcal{D})$ with $|\theta_I| = |\varepsilon_I| = 1$, $I \in \mathcal{D}$ such that

$$\sup_{\|c\|_{\ell^\infty} \leq 1} |\langle f, r_m^{(c)} \rangle| = |\langle f, r_m^{(\theta)} \rangle| \quad \text{and} \quad \sup_{\|c\|_{\ell^\infty} \leq 1} |\langle Tr_m^{(c)}, g \rangle| = |\langle Tr_m^{(\varepsilon)}, g \rangle|. \quad (4.3)$$

for all $m \in \mathbb{N}$.

Now let $(\omega_m)_{m=1}^M$ denote a finite sequence of scalars and consider that by (1.4) we have

$$\sum_{m=1}^M \omega_m \langle f, r_m^{(\theta)} \rangle \leq \|f\|_{SL^\infty} \left\| \sum_{m=1}^M \omega_m r_m^{(\theta)} \right\|_{H^1} \leq \|f\|_{SL^\infty} \left(\sum_{m=1}^M \omega_m^2 \right)^{1/2}.$$

Putting $\omega_m = \langle f, r_m^{(\theta)} \rangle$ gives

$$\left(\sum_{m=1}^M |\langle f, r_m^{(\theta)} \rangle|^2 \right)^{1/2} \leq \|f\|_{SL^\infty}.$$

Combining the latter estimate with (4.3) yields the first part of (4.2).

The argument for the second part is similar. By (1.4), we obtain that

$$\sum_{m=1}^M \omega_m \langle Tr_m^{(\varepsilon)}, g \rangle \leq \|T\| \|g\|_{H^1} \left\| \sum_{m=1}^M \omega_m r_m^{(\varepsilon)} \right\|_{SL^\infty} \leq \|T\| \|g\|_{H^1} \left(\sum_{m=1}^M \omega_m^2 \right)^{1/2}.$$

By putting $\omega_m = \langle Tr_m^{(\varepsilon)}, g \rangle$ we obtain

$$\left(\sum_{m=1}^M |\langle Tr_m^{(\varepsilon)}, g \rangle|^2 \right)^{1/2} \leq \|T\| \|g\|_{H^1},$$

which when combined with (4.3) concludes the proof. \square

Here we come to the crucial Lemma that enables infinite dimensional reasoning in the non-separable Banach space SL^∞ .

Lemma 4.2. *Let $\eta > 0$, $g \in H^1$ and let $\Gamma \subset \mathbb{N}$ be infinite. Suppose that $T : SL^\infty \rightarrow SL^\infty$ is a bounded linear operator. Then there exists an infinite set $\Lambda \subset \Gamma$ such that*

$$\sup_{\|f\|_{SL^\infty} \leq 1} |\langle TP_\Lambda f, g \rangle| \leq \eta \|g\|_{H^1},$$

where the norm one projection $P_\Lambda : SL^\infty \rightarrow SL^\infty$ is given by

$$P_\Lambda \left(\sum_{I \in \mathcal{D}} a_I h_I \right) = \sum_{m \in \Lambda} \sum_{I \in \mathcal{D}_m} a_I h_I.$$

Proof of Lemma 4.2. Let $\eta > 0$ and $g \in H^1$ and assume that $\Gamma = \mathbb{N}$. Suppose the conclusion of the Lemma is false. Define $k = \lceil \frac{\|T\|^2}{\eta^2} \rceil$ and choose infinite, disjoint sets $\Lambda_1, \Lambda_2, \dots, \Lambda_k$. By our assumption, we can find $f_1, f_2, \dots, f_k \in SL^\infty$ with $\|f_j\|_{SL^\infty} = 1$, $1 \leq j \leq k$ such that

$$\langle TP_{\Lambda_j} f_j, g \rangle > \eta \|g\|_{H^1}, \quad \text{for all } 1 \leq j \leq k.$$

Summing these estimates and using (1.4) yields

$$\|g\|_{H^1} \|T\| \left\| \sum_{j=1}^k P_{\Lambda_j} f_j \right\|_{SL^\infty} > k\eta \|g\|_{H^1}. \quad (4.4)$$

Since the Λ_j , $1 \leq j \leq k$ are disjoint, we have that

$$\mathbb{S} \left(\sum_{j=1}^k P_{\Lambda_j} f_j \right) = \left(\sum_{j=1}^k \mathbb{S}(P_{\Lambda_j} f_j)^2 \right)^{1/2},$$

and therefore we obtain

$$\left\| \sum_{j=1}^k P_{\Lambda_j} f_j \right\|_{SL^\infty} \leq \left(\sum_{j=1}^k \|P_{\Lambda_j} f_j\|_{SL^\infty}^2 \right)^{1/2} \leq k^{1/2}. \quad (4.5)$$

By combining the estimates (4.4) and (4.5), we reach a contradiction. \square

4.2. Diagonalization of operators on SL^∞ .

We will show that any given operator T acting on SL^∞ with large diagonal can be almost-diagonalized by a block basis of the Haar system $(b_I : I \in \mathcal{D})$, that spans a complemented copy of SL^∞ (see Theorem 4.3). This is achieved by constructing $(b_I : I \in \mathcal{D})$ with aid from the results in Section 4.1, so that Jones' compatibility conditions (J1)–(J4) are satisfied.

From here on, we will regularly identify a dyadic interval $I \in \mathcal{D}$ with its natural ordering number $\mathcal{O}(I)$ given by

$$\mathcal{O}(I) = 2^n - 1 + k, \quad \text{if } I = [k2^{-n}, (k+1)2^{-n}].$$

To be precise, for $\mathcal{O}(I) = i$ we identify

$$\mathcal{B}_I = \mathcal{B}_i \quad \text{and} \quad b_I^{(\varepsilon)} = b_i^{(\varepsilon)}.$$

The block basis $(b_I^{(\varepsilon)} : I \in \mathcal{D})$ is defined in (3.5). See also below.

Theorem 4.3. *Let $\delta \geq 0$ and let $T : SL^\infty \rightarrow SL^\infty$ be an operator satisfying*

$$|\langle Th_I, h_I \rangle| \geq \delta |I|, \quad I \in \mathcal{D}.$$

Then for any $\eta > 0$, there exists a sequence of collections $(\mathcal{B}_I : I \in \mathcal{D})$ and a sequence of signs $\varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{D})$ which generate the block basis of the Haar system $(b_I^{(\varepsilon)} : I \in \mathcal{D})$ given by

$$b_I^{(\varepsilon)} = \sum_{K \in \mathcal{B}_I} \varepsilon_K h_K, \quad I \in \mathcal{D},$$

so that the following conditions are satisfied:

- (i) $(\mathcal{B}_I : I \in \mathcal{D})$ satisfies Jones' compatibility conditions (J1)–(J4) with constant $\kappa_J = 1$.
- (ii) $(b_I^{(\varepsilon)} : I \in \mathcal{D})$ almost-diagonalizes T so that T has large diagonal with respect to $(b_I^{(\varepsilon)} : I \in \mathcal{D})$. To be more precise, for any $i \in \mathbb{N}_0$ we have the estimates

$$\sum_{j=0}^{i-1} |\langle Tb_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle| \leq \eta 4^{-i} \|b_i^{(\varepsilon)}\|_2^2, \quad (4.6a)$$

$$\langle Tb_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle \geq \delta \|b_i^{(\varepsilon)}\|_2^2, \quad (4.6b)$$

$$\sup \left\{ |\langle Tg, b_i^{(\varepsilon)} \rangle| : g = \sum_{j=i+1}^{\infty} a_j b_j^{(\varepsilon)}, \|g\|_{SL^\infty} \leq 1 \right\} \leq \eta 4^{-i} \|b_i^{(\varepsilon)}\|_2^2. \quad (4.6c)$$

Proof of Theorem 4.3.

Let $\delta \geq 0$, $\eta > 0$ and $T : SL^\infty \rightarrow SL^\infty$. Before we begin with the actual proof, observe that by 1-unconditionality, we can assume that

$$\langle Th_I, h_I \rangle \geq \delta |I|, \quad I \in \mathcal{D}.$$

Given $I \in \mathcal{D}$, we write

$$Th_I = \alpha_I h_I + r_I, \quad (4.7a)$$

where

$$\alpha_I = \frac{\langle Th_I, h_I \rangle}{|I|} \quad \text{and} \quad r_I = \sum_{J \neq I} \frac{\langle Th_I, h_J \rangle}{|J|} h_J. \quad (4.7b)$$

We note the estimate

$$\delta \leq \alpha_I \leq \|T\|. \quad (4.8)$$

Inductive construction of $(b_I^{(\varepsilon)} : I \in \mathcal{D})$.

To begin the induction, we simply put

$$\mathcal{B}_0 = \mathcal{B}_{[0,1)} = \{[0,1)\} \quad \text{and} \quad b_0^{(\varepsilon)} = b_{[0,1)}^{(\varepsilon)} = h_{[0,1)}.$$

We complete the initial step of our construction, by choosing $\Lambda_1 \subset \mathbb{N}$ according to Lemma 4.2 such that

$$\sup_{\|f\|_{SL^\infty} \leq 1} |\langle TP_{\Lambda_1} f, b_0^{(\varepsilon)} \rangle| \leq \eta \|b_0^{(\varepsilon)}\|_2^2.$$

For the inductive step, let us now assume that we have already

- ▷ chosen a strictly increasing sequence of integers (m_j) and infinite index sets $\Lambda_1 \supset \cdots \supset \Lambda_i$ with $m_j \in \Lambda_j \setminus \Lambda_{j+1}$, $1 \leq j \leq i-1$,
- ▷ constructed finite collections \mathcal{B}_j with $\mathcal{B}_j \subset \mathcal{D}_{m_j}$, $0 \leq j \leq i-1$,
- ▷ made a suitable choice of signs $\varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \bigcup_{j < i} \mathcal{B}_j)$,
- ▷ and the block basis elements $b_j^{(\varepsilon)}$ have the form

$$b_j^{(\varepsilon)} = \sum_{K \in \mathcal{B}_j} \varepsilon_K h_K, \quad 0 \leq j \leq i-1.$$

We will now choose an integer $m_i \in \Lambda_i$ with $m_i > m_{i-1}$, construct a finite collection $\mathcal{B}_i \subset \mathcal{D}_{m_i}$, choose signs $\varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{B}_i)$, select an infinite subset $\Lambda_{i+1} \subset \Lambda_i \setminus \{m_i\}$ and define $b_i^{(\varepsilon)}$ by

$$b_i^{(\varepsilon)} = \sum_{K \in \mathcal{B}_i} \varepsilon_K h_K, \quad (4.9a)$$

such that the operator T is almost-diagonalized by the block basis $(b_i^{(\varepsilon)})$ while preserving the large diagonal. To be precise:

$$\sum_{j=0}^{i-1} |\langle T b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle| \leq \eta 4^{-i} \|b_i^{(\varepsilon)}\|_2^2, \quad (4.9b)$$

$$\langle T b_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle \geq \delta \|b_i^{(\varepsilon)}\|_2^2, \quad (4.9c)$$

$$\sup_{\|f\|_{SL^\infty} \leq 1} |\langle TP_{\Lambda_{i+1}} f, b_i^{(\varepsilon)} \rangle| \leq \eta 4^{-i} \|b_i^{(\varepsilon)}\|_2^2. \quad (4.9d)$$

For a definition of the projection P_Λ see Lemma 4.2. For the most part of this inductive construction step, we will assume that $\Lambda_i = \mathbb{N}$.

Now, let $I \in \mathcal{D}$ be such that $\mathcal{O}(I) = i$. The dyadic interval \tilde{I} denotes the unique dyadic interval such that $\tilde{I} \supset I$ and $|\tilde{I}| = 2|I|$. Furthermore, for every dyadic



FIGURE 1. The picture shows the construction of \mathcal{F}_m , if I is the left half of \tilde{I} . The large dyadic intervals K_0 on top form the set $\mathcal{B}_{\tilde{I}}$. The medium sized dyadic intervals K_0^ℓ denote the left half of the K_0 , and the set $B_{\tilde{I}}^\ell$ is the union of the K_0^ℓ . The small intervals K at the bottom form the high-frequency cover \mathcal{F}_m of the set $B_{\tilde{I}}^\ell$.

interval K_0 , we denote its left half by K_0^ℓ and its right half by K_0^r . Following the construction of Gamlen-Gaudet [12], we define the sets

$$B_{\tilde{I}}^\ell = \bigcup \{K_0^\ell : K_0 \in \mathcal{B}_{\tilde{I}}\} \quad \text{and} \quad B_{\tilde{I}}^r = \bigcup \{K_0^r : K_0 \in \mathcal{B}_{\tilde{I}}\}.$$

If I is the *left half* of \tilde{I} we put

$$\mathcal{F}_m = \{K \in \mathcal{D} : |K| = 2^{-m}, K \subset B_{\tilde{I}}^\ell\}. \quad (4.10a)$$

If I is the *right half* of \tilde{I} we define

$$\mathcal{F}_m = \{K \in \mathcal{D} : |K| = 2^{-m}, K \subset B_{\tilde{I}}^r\}. \quad (4.10b)$$

See Figure 1 for a depiction of \mathcal{F}_m . In either of the cases (4.10a) and (4.10b) we put

$$f_m^{(\varepsilon)} = \sum_{K \in \mathcal{F}_m} \varepsilon_K h_K, \quad (4.11)$$

for all $m \in \mathbb{N}$ and $\varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{F}_m)$.

Choosing the frequency m_i .

Note that by (4.10) and our induction hypothesis we have $\mathcal{F}_m \cap \mathcal{B}_j = \emptyset$, $0 \leq j \leq i-1$, $m \in \mathbb{N}$. In particular, the sequence $(\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{F}_m)$ does not interfere with any of the previous definitions of $b_j^{(\varepsilon)}$, $0 \leq j \leq i-1$. By Lemma 4.1, we have that

$$\lim_{m \rightarrow \infty} \sup \{|\langle f, f_m^{(\varepsilon)} \rangle| : \varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{F}_m)\} = 0, \quad f \in SL^\infty.$$

consequently, we obtain the estimate

$$\sup \left\{ \sum_{j=0}^{i-1} |\langle T b_j^{(\varepsilon)}, f_{m_i}^{(\varepsilon)} \rangle| : \varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{F}_{m_i}) \right\} \leq \eta 4^{-i} \|f_{m_i}^{(\varepsilon)}\|_2^2, \quad (4.12)$$

for sufficiently large m_i . Certainly, we choose m_i large enough so that $\mathcal{F}_{m_i} \neq \emptyset$, see (4.10). Note that $\|f_{m_i}^{(\varepsilon)}\|_2^2 = |I|$.

Choosing the signs ε .

Continuing with the proof, we obtain from (4.7) that

$$T f_{m_i}^{(\varepsilon)} = \sum_{K \in \mathcal{F}_{m_i}} \varepsilon_K \alpha_K h_K + R_{m_i}^{(\varepsilon)}, \quad (4.13)$$

where

$$R_{m_i}^{(\varepsilon)} = \sum_{K \in \mathcal{F}_{m_i}} \varepsilon_K r_K. \quad (4.14)$$

For all $\varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{F}_{m_i})$, we define

$$X_{m_i}(\varepsilon) = \langle R_{m_i}^{(\varepsilon)}, f_{m_i}^{(\varepsilon)} \rangle.$$

From (4.13) and (4.8) follows that

$$\langle T f_{m_i}^{(\varepsilon)}, f_{m_i}^{(\varepsilon)} \rangle \geq \delta \|f_{m_i}^{(\varepsilon)}\|_2^2 + X_{m_i}(\varepsilon). \quad (4.15)$$

By (4.7) we have that $\langle r_K, h_K \rangle = 0$, hence

$$X_{m_i}(\varepsilon) = \sum_{\substack{K_0, K_1 \in \mathcal{F}_{m_i} \\ K_0 \neq K_1}} \varepsilon_{K_0} \varepsilon_{K_1} \langle r_{K_0}, h_{K_1} \rangle.$$

Now, let \mathbb{E}_ε denote the averaging over all possible choices of signs $\varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{F}_{m_i})$. If $K_0 \neq K_1$, then $\mathbb{E}_\varepsilon \varepsilon_{K_0} \varepsilon_{K_1} = 0$ and therefore

$$\mathbb{E}_\varepsilon X_{m_i} = 0.$$

Taking the expectation \mathbb{E}_ε in (4.15) and considering the above identity, we obtain

$$\mathbb{E}_\varepsilon \langle T f_{m_i}^{(\varepsilon)}, f_{m_i}^{(\varepsilon)} \rangle \geq \delta \|f_{m_i}^{(\varepsilon)}\|_2^2. \quad (4.16)$$

The expectation on the right hand side is not present since $\|f_{m_i}^{(\varepsilon)}\|_2^2 = |I|$ for all choices of ε . By (4.16) we can find an $\varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{F}_{m_i})$ such that

$$\langle T f_{m_i}^{(\varepsilon)}, f_{m_i}^{(\varepsilon)} \rangle \geq \delta \|f_{m_i}^{(\varepsilon)}\|_2^2. \quad (4.17)$$

Choosing the set Λ_{i+1} .

The next step is to find an infinite set $\Lambda_{i+1} \subset \Lambda_i \setminus \{m_i\}$ such that (4.9d) is satisfied. To this end, we apply Lemma 4.2, to the infinite set Γ given by

$$\Gamma = \{n \in \Lambda_i : n > m_i\} \subset \Lambda_i.$$

Thus, we obtain $\Lambda_{i+1} \subset \Gamma$ such that

$$\sup_{\|f\|_{SL^\infty} \leq 1} |\langle T P_{\Lambda_{i+1}} f, f_{m_i}^{(\varepsilon)} \rangle| \leq \eta 4^{-i} \|f_{m_i}^{(\varepsilon)}\|_2^2. \quad (4.18)$$

We conclude the inductive construction step by defining

$$\mathcal{B}_i = \mathcal{B}_I = \mathcal{F}_{m_i} \quad \text{and} \quad b_i^{(\varepsilon)} = b_I^{(\varepsilon)} = f_{m_i}^{(\varepsilon)}. \quad (4.19)$$

Conclusion.

We remark that we chose m_i and ε according to (4.12) and (4.17), which together with (4.18) shows Theorem 4.3 (ii). It follows immediately from the Gamlen-Gaudet construction [12] of $(\mathcal{B}_I : I \in \mathcal{D})$ that the collection $(\mathcal{B}_I : I \in \mathcal{D})$ satisfies Jones' compatibility conditions (J1)–(J4) with $\kappa_J = 1$. \square

4.3. Factorization in SL^∞ – Proof of Theorem 2.1.

We use the almost-diagonalization result in Section 4.2 to prove the main result Theorem 2.1.

Let $\delta, \eta > 0$, and let $T : SL^\infty \rightarrow SL^\infty$ be an operator satisfying

$$|\langle T h_I, h_I \rangle| \geq \delta |I|, \quad I \in \mathcal{D}.$$

Let $\eta' = \eta(\delta, \eta)$ denote a constant so that

$$\frac{4\eta'}{\delta} < 1 \quad \text{and} \quad \frac{1}{1 - \frac{4\eta'}{\delta}} \leq 1 + \eta. \quad (4.20)$$

By Theorem 4.3, we obtain a sequence of collections $(\mathcal{B}_I : I \in \mathcal{D})$ and a sequence of signs $\varepsilon = (\varepsilon_K : \varepsilon_K \in \{\pm 1\}, K \in \mathcal{D})$ which generate the block basis of the Haar system $(b_I^{(\varepsilon)} : I \in \mathcal{D})$ given by

$$b_I^{(\varepsilon)} = \sum_{K \in \mathcal{B}_I} \varepsilon_K h_K, \quad I \in \mathcal{D},$$

so that the following conditions are satisfied:

- (i) $(\mathcal{B}_I : I \in \mathcal{D})$ satisfies Jones' compatibility conditions (J1)–(J4) with constant $\kappa_J = 1$.
- (ii) For all $i \in \mathbb{N}_0$ we have the estimates

$$\sum_{j=0}^{i-1} |\langle Tb_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle| \leq \eta' 4^{-i} \|b_i^{(\varepsilon)}\|_2^2, \quad (4.21a)$$

$$\langle Tb_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle \geq \delta \|b_i^{(\varepsilon)}\|_2^2, \quad (4.21b)$$

$$\sup \left\{ |\langle Tg, b_i^{(\varepsilon)} \rangle| : g = \sum_{j=i+1}^{\infty} a_j b_j^{(\varepsilon)}, \|g\|_{SL^\infty} \leq 1 \right\} \leq \eta' 4^{-i} \|b_i^{(\varepsilon)}\|_2^2. \quad (4.21c)$$

Since $(\mathcal{B}_I : I \in \mathcal{D})$ satisfies Jones' compatibility conditions (J1)–(J4) with $\kappa_J = 1$, Remark 3.5 and Theorem 3.3 imply that the operators

$$B^{(\varepsilon)} f = \sum_{I \in \mathcal{J}} \frac{\langle f, h_I \rangle}{\|h_I\|_2^2} b_I^{(\varepsilon)} \quad \text{and} \quad Q^{(\varepsilon)} f = \sum_{I \in \mathcal{J}} \frac{\langle f, b_I^{(\varepsilon)} \rangle}{\|b_I^{(\varepsilon)}\|_2^2} h_I \quad (4.22)$$

satisfy the estimates

$$\|B^{(\varepsilon)} f\|_{SL^\infty} \leq \|f\|_{SL^\infty} \quad \text{and} \quad \|Q^{(\varepsilon)} f\|_{SL^\infty} \leq \|f\|_{SL^\infty}. \quad (4.23)$$

By (4.23), the operator $P^{(\varepsilon)} : SL^\infty \rightarrow SL^\infty$ defined as $P^{(\varepsilon)} = B^{(\varepsilon)} Q^{(\varepsilon)}$, is given by

$$P^{(\varepsilon)} f = \sum_{I \in \mathcal{J}} \frac{\langle f, b_I^{(\varepsilon)} \rangle}{\|b_I^{(\varepsilon)}\|_2^2} b_I^{(\varepsilon)}, \quad f \in SL^\infty. \quad (4.24)$$

Therefore, $P^{(\varepsilon)}$ is an orthogonal projection with the estimate

$$\|P^{(\varepsilon)} f\|_{SL^\infty} \leq \|f\|_{SL^\infty}, \quad f \in SL^\infty. \quad (4.25)$$

Let Y denote the subspace of SL^∞ given by

$$Y = \left\{ g = \sum_{i=0}^{\infty} a_i b_i^{(\varepsilon)} : a_i \in \mathbb{R}, \|g\|_{SL^\infty} < \infty \right\}.$$

Note the following commutative diagram:

$$\begin{array}{ccc} SL^\infty & \xrightarrow{\text{Id}} & SL^\infty \\ B^{(\varepsilon)} \downarrow & & \uparrow B^{(\varepsilon)-1} \\ Y & \xrightarrow{\text{Id}} & Y \end{array} \quad \|B^{(\varepsilon)}\|, \|B^{(\varepsilon)-1}\| \leq 1. \quad (4.26)$$

The estimates for $\|B^{(\varepsilon)}\|, \|B^{(\varepsilon)-1}\|$ follow from (4.23). Now, define $U : SL^\infty \rightarrow Y$ by

$$Uf = \sum_{i=0}^{\infty} \frac{\langle f, b_i^{(\varepsilon)} \rangle}{\langle Tb_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle} b_i^{(\varepsilon)}, \quad (4.27)$$

and note that by (4.9c), the 1-unconditionality of the Haar system in SL^∞ and (4.25), the operator U has the upper bound

$$\|U : SL^\infty \rightarrow Y\|_{SL^\infty} \leq \frac{1}{\delta}. \quad (4.28)$$

Observe that for all $g = \sum_{i=0}^{\infty} a_i b_i^{(\varepsilon)} \in Y$ the following identity is true:

$$UTg - g = \sum_{i=0}^{\infty} \left(\sum_{j:j < i} a_j \frac{\langle Tb_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle}{\langle Tb_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle} + \frac{\langle T \sum_{j:j > i} a_j b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle}{\langle Tb_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle} \right) b_i^{(\varepsilon)}. \quad (4.29)$$

Noting that $|a_j| \leq \|g\|_{SL^\infty}$ and using the estimates (4.9) yields

$$\|UTg - g\|_{SL^\infty} \leq \frac{4\eta'}{\delta} \|g\|_{SL^\infty}. \quad (4.30)$$

Finally, let $J : Y \rightarrow SL^\infty$ denote the operator given by $Jy = y$. By our choice (4.20) the operator $V : SL^\infty \rightarrow Y$ given by $V = (UTJ)^{-1}U$ is well defined and

$$\begin{array}{ccc} Y & \xrightarrow{\text{Id}} & Y \\ & \searrow (UTJ)^{-1} & \nearrow \\ & Y & \\ J \downarrow & & \uparrow V \\ SL^\infty & \xrightarrow{T} & SL^\infty \end{array} \quad \|J\| \|V\| \leq (1 + \eta)/\delta. \quad (4.31)$$

Merging the commutative diagrams (4.26) and (4.31) concludes the proof. \square

Remark 4.4. We remark that in the identity (4.29) above, the non-separability of SL^∞ prevents us from expanding $\langle T \sum_{j:j>i} a_j b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle$ into $\sum_{j:j>i} a_j \langle T b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle$. Therefore, in passing from (4.29) to (4.30), we have to estimate the infinite sum $\langle T \sum_{j:j>i} a_j b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle$ directly. This is achieved by Lemma 4.2, which results in estimate (4.9d).

4.4. SL^∞ is primary – Proof of Theorem 2.2.

Here we prove the second main result Theorem 2.2 and show that SL^∞ is primary. Since the following proof has been given in numerous situations, see e.g. [24] we will only describe its major steps.

- ▷ Diagonalization of T by Theorem 4.3 with parameter $\delta = 0$ yields a block basis $(b_i : i \in \mathbb{N}_0)$ such that

$$\begin{aligned} \sum_{j=0}^{i-1} |\langle T b_i, b_j \rangle| &\leq \eta 4^{-i} \|b_i\|_2^2, \\ |\langle T \sum_{j:j>i} a_j b_j, b_i \rangle| &\leq \eta 4^{-i} \|b_i\|_2^2 \left\| \sum_{j:j>i} a_j b_j \right\|_{SL^\infty}. \end{aligned}$$

- ▷ Finding a “large” subcollection of dyadic intervals in one of the following collections:

$$\{I \in \mathcal{D} : \langle T b_I, b_I \rangle \geq \|b_I\|_2^2/2\} \quad \text{or} \quad \{I \in \mathcal{D} : \langle (\text{Id} - T)b_I, b_I \rangle \geq \|b_I\|_2^2/2\}.$$

It is well established how to construct a sequence of collections $(\mathcal{C}_I : I \in \mathcal{D})$ either entirely inside the first collection, or entirely inside the second collection, such that Jones’ compatibility conditions (J1)–(J4) are satisfied. We refer the reader to [12]. See also [24].

- ▷ Using the reiteration Theorem 3.2 and the projection Theorem 3.3 with parameter $\delta = 1/2$, we obtain a block basis $(c_I^{(\varepsilon)} : I \in \mathcal{D})$ of the Haar system given by

$$c_I^{(\varepsilon)} = \sum_{K \in \mathcal{C}_I} b_K^{(\varepsilon)} = \sum_{K \in \mathcal{C}_I} \sum_{Q \in \mathcal{B}_K} \varepsilon_K h_K,$$

so that $(c_I^{(\varepsilon)} : I \in \mathcal{D})$ is 1-equivalent to $(h_I : I \in \mathcal{D})$, and the subspace Y of SL^∞ defined by

$$Y = \left\{ g = \sum_{I \in \mathcal{D}} a_I c_I^{(\varepsilon)} : a_i \in \mathbb{R}, \|g\|_{SL^\infty} < \infty \right\}.$$

is complemented in SL^∞ . The projection onto Y can be chosen with norm $\leq 2 + \eta$.

- ▷ The rest of the proof is repeating the argument in the proof of Theorem 3.3 (with $c_I^{(\varepsilon)}$ taking the place of $b_I^{(\varepsilon)}$) to obtain that

$$\begin{array}{ccc} SL^\infty & \xrightarrow{\text{Id}} & SL^\infty \\ R \downarrow & & \uparrow S \\ SL^\infty & \xrightarrow{T} & SL^\infty \end{array} \quad \|R\|\|S\| \leq 2 + \eta.$$

- ▷ Finally, consider the collections $\mathcal{A}_k \subset \mathcal{D}$ given by

$$\mathcal{A}_k = \{I \in \mathcal{D} : I \subset [1 - 2^{-k-1}, 1 - 2^{-k}]\}, \quad k \in \mathbb{N},$$

and note that with the obvious isomorphism we obtain that

$$SL^\infty = \left(\sum SL^\infty \right)_\infty.$$

Thus, Pełczyński's decomposition method and the above factorization diagram imply the primarity of SL^∞ . \square

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